

On complete hypersurfaces with constant mean and scalar curvatures in Euclidean spaces

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Abstract. Generalizing a theorem of Huang, Cheng and Wan classified the complete hypersurfaces of \mathbb{R}^4 with non-zero constant mean curvature and constant scalar curvature. In our work, we obtain results of this nature in higher dimensions. In particular, we prove that if a complete hypersurface of \mathbb{R}^5 has constant mean curvature $H \neq 0$ and constant scalar curvature $R \geq \frac{2}{3}H^2$, then $R = H^2$, $R = \frac{8}{9}H^2$ or $R = \frac{2}{3}H^2$. Moreover, we characterize the hypersurface in the cases $R = H^2$ and $R = \frac{8}{9}H^2$, and provide an example in the case $R = \frac{2}{3}H^2$. The proofs are based on the principal curvature theorem of Smyth-Xavier and a well known formula for the Laplacian of the squared norm of the second fundamental form of a hypersurface in a space form.

1 Introduction

A well known result of Klotz and Osserman [7] states that the round spheres and the circular cylinders are the only complete surfaces in the Euclidean 3-space \mathbb{R}^3 with non-zero constant mean curvature whose Gaussian curvature does not change sign. In higher dimensions, Nomizu and Smyth [10] proved the following result: *If M^n is a complete Riemannian manifold with non-negative sectional curvature and constant scalar curvature, and $f : M^n \rightarrow \mathbb{R}^{n+1}$*

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is an isometric immersion with constant mean curvature, then $f(M^n)$ is a generalized cylinder $\mathbb{R}^{n-k} \times \mathbb{S}^k$, for some $k = 1, \dots, n$.

Cheng and Yau proved in [3] that the above mentioned result of Nomizu and Smyth holds without the assumption that the scalar curvature is constant, and in [4] that it holds without the assumption that the mean curvature is constant. Extending these results of Cheng and Yau, Hartman proved in [5] that if M^n is a complete Riemannian manifold with non-negative sectional curvature and $f : M^n \rightarrow \mathbb{R}^{n+1}$ is an isometric immersion with positive constant r -th mean curvature H_r , for some $r = 1, \dots, n$, then $f(M^n) = \mathbb{R}^{n-d} \times \mathbb{S}^d$, for some $r \leq d \leq n$.

In view of the above discussion, a question that arises naturally is whether the theorem of Nomizu-Smyth mentioned in the first paragraph holds without the assumption that the sectional curvatures of M^n are non-negative. For $n = 3$, a partial answer to this question was given by Huang [6]:

Teorema (Huang). *Let M^3 be a complete and connected Riemannian manifold with non-negative constant scalar curvature R , and let $f : M^3 \rightarrow \mathbb{R}^4$ be an isometric immersion with non-zero constant mean curvature H . Then, $R = H^2$, $R = \frac{3}{4}H^2$ or $R = 0$. When $R = H^2$, $f(M^3) = \mathbb{S}^3(\frac{1}{|H|})$, and when $R = \frac{3}{4}H^2$, $f(M^3) = \mathbb{R} \times \mathbb{S}^2(\frac{2}{3|H|})$.*

Later, Cheng and Wan [2] improved the above theorem of Huang in two aspects. They not only showed that the hypothesis $R \geq 0$ is superfluous, but also characterized the hypersurface in the case $R = 0$. More precisely, they obtained the following result.

Teorema (Cheng-Wan). *Let M^3 be a complete and connected Riemannian manifold with constant scalar curvature R , and let $f : M^3 \rightarrow \mathbb{R}^4$ be an isometric immersion with non-zero constant mean curvature H . Then $R = H^2$, $R = \frac{3}{4}H^2$ or $R = 0$. When $R = H^2$, $f(M^3) = \mathbb{S}^3(\frac{1}{|H|})$; when $R = \frac{3}{4}H^2$, $f(M^3) = \mathbb{R} \times \mathbb{S}^2(\frac{2}{3|H|})$ and when $R = 0$, $f(M^3) = \mathbb{R}^2 \times \mathbb{S}^1(\frac{1}{3|H|})$.*

In the same spirit of the results of Huang and Cheng-Wan referred to above, we establish in this work the following theorems:

Theorem 1.1. *Let M^4 be a complete and connected Riemannian manifold with constant scalar curvature R , and $f : M^4 \rightarrow \mathbb{R}^5$ an isometric immersion with non-zero constant mean curvature H . If $R \geq \frac{2}{3}H^2$ then $R = H^2$, $R = \frac{8}{9}H^2$ or $R = \frac{2}{3}H^2$. When $R = H^2$, $f(M^4) = \mathbb{S}^4(\frac{1}{|H|})$ and when $R = \frac{8}{9}H^2$, $f(M^4) = \mathbb{R} \times \mathbb{S}^3(\frac{3}{4|H|})$.*

Remark 1. When $R = \frac{2}{3}H^2$, one has as an example the cylinder $\mathbb{R}^2 \times \mathbb{S}^2(\frac{1}{2|H|})$.

Theorem 1.2. Let M^5 be a complete and connected Riemannian manifold with constant scalar curvature R , and $f : M^5 \rightarrow \mathbb{R}^6$ an isometric immersion with non-zero constant mean curvature H and non-negative 4-th mean curvature H_4 . If $R \geq \frac{5}{8}H^2$ then $R = H^2$, $R = \frac{15}{16}H^2$, $R = \frac{5}{6}H^2$ or $R = \frac{5}{8}H^2$. When $R = H^2$, $f(M^5) = \mathbb{S}^5(\frac{1}{|H|})$ and when $R = \frac{15}{16}H^2$, $f(M^5) = \mathbb{R} \times \mathbb{S}^4(\frac{4}{5|H|})$.

Remark 2. When $R = \frac{5}{6}H^2$, one has as an example the cylinder $\mathbb{R}^2 \times \mathbb{S}^3(\frac{3}{5|H|})$ and when $R = \frac{5}{8}H^2$, the cylinder $\mathbb{R}^3 \times \mathbb{S}^2(\frac{2}{5|H|})$.

Theorem 1.3. Let M^n be a complete and connected Riemannian manifold of dimension $n \geq 3$ and constant scalar curvature R , and let $f : M^n \rightarrow \mathbb{R}^{n+1}$ be an isometric immersion with non-zero constant mean curvature H . If $HH_3 \geq 0$ and $0 \leq R \leq \frac{nH^2}{2(n-1)}$, then $R = 0$ or $R = \frac{nH^2}{2(n-1)}$. In case that $R = 0$, $f(M^n) = \mathbb{R}^{n-1} \times \mathbb{S}^1(\frac{1}{n|H|})$ and in case that $R = \frac{nH^2}{2(n-1)}$, $f(M^n) = \mathbb{R}^{n-2} \times \mathbb{S}^2(\frac{2}{n|H|})$.

Remark 3. If it were possible to remove the hypothesis $HH_3 \geq 0$ in Theorem 1.3, then a combination of this theorem (for $n = 4$) with Theorem 1.1 would provide an extension of the theorem of Cheng-Wan for hypersurfaces with non-negative scalar curvatures in \mathbb{R}^5 .

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2 Preliminaries

Given an isometric immersion $f : M^n \rightarrow N_c^{n+1}$ of an orientable n -dimensional Riemannian manifold M^n into an orientable $(n+1)$ -dimensional Riemannian manifold N_c^{n+1} of constant sectional curvature c , we denote by A the shape operator of f with respect to a global unit normal vector field ξ , and by $\lambda_1, \dots, \lambda_n$ the eigenvalues of A (the principal curvatures of M^n). It is well known that if we label the principal curvatures at each point by the condition $\lambda_1 \leq \dots \leq \lambda_n$, then the principal curvature functions $\lambda_i : M \rightarrow \mathbb{R}$, $i = 1, \dots, n$, become continuous.

The r -th mean curvature H_r , $1 \leq r \leq n$, of the immersion is defined by

$$\binom{n}{r} H_r = S_r := \sum_{i_1 < \dots < i_r} \lambda_{i_1} \dots \lambda_{i_r}. \quad (2.1)$$

Notice that H_1 is the mean curvature H and $H_n = \lambda_1 \lambda_2 \dots \lambda_n$ is the Gauss-Kronecker curvature of the immersion. The function H_2 is up to a constant the (normalized) scalar curvature R of M^n . Indeed, if for a given $p \in M$ we consider an orthonormal basis $\{e_1, \dots, e_n\}$ of $T_p M$ that diagonalizes A , then the sectional curvature $K(e_i, e_j)$ of the plane spanned by e_i and e_j is, by the Gauss equation, given by

$$K(e_i, e_j) = c + \lambda_i \lambda_j, \quad (2.2)$$

and so

$$R = \frac{1}{\binom{n}{2}} \sum_{i < j} K(e_i, e_j) = \frac{1}{\binom{n}{2}} \sum_{i < j} (c + \lambda_i \lambda_j) = c + H_2. \quad (2.3)$$

For what follows it is convenient to write the r -th mean curvature $H_r(p)$ of M^n at a point p as

$$\binom{n}{r} H_r(p) = S_r(p) = \sigma_r(\lambda_1(p), \dots, \lambda_n(p)), \quad (2.4)$$

where σ_r is the r -th symmetric function on \mathbb{R}^n ,

$$\sigma_r(x_1, \dots, x_n) = \sum_{i_1 < \dots < i_r} x_{i_1} \dots x_{i_r}. \quad (2.5)$$

For future use, we observe that

$$\sigma_r(x) = x_i \sigma_{r-1}(\widehat{x}_i) + \sigma_r(\widehat{x}_i), \quad i, r = 1, \dots, n, \quad x \in \mathbb{R}^n, \quad (2.6)$$

where $\sigma_0 = 1$ and, for instance, $\sigma_{r-1}(\widehat{x}_i) = \sigma_{r-1}(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$.

The squared norm $|A|^2$ of the shape operator A is defined as the trace of A^2 . Taking an orthonormal basis that diagonalizes A , it is easy to see that

$$|A|^2 = \sum_i \lambda_i^2. \quad (2.7)$$

Hence, from (2.1), (2.3) and (2.7), we obtain

$$n^2 H^2 = \left(\sum_{i=1}^n \lambda_i \right)^2 = \sum_{i=1}^n \lambda_i^2 + \sum_{i \neq j} \lambda_i \lambda_j = |A|^2 + n(n-1)(R-c). \quad (2.8)$$

We will finish this section recalling the definition of the Newton's tensors P_r , $0 \leq r \leq n$, associated to the shape operator A of an isometric immersion. These tensors are defined inductively by

$$\begin{aligned} P_0 &= I, \\ P_r &= S_r I - A P_{r-1}, \quad 1 \leq r \leq n. \end{aligned} \quad (2.9)$$

3 Main ingredients

In this section we will present the basic tools for the proofs of our results. We begin stating the following proposition [9, p. 668], which provides a formula for the Laplacian of $|A|^2$ for a hypersurface in a space of constant curvature.

Proposition 3.1. *Let $f : M^n \rightarrow N_c^{n+1}$ be an isometric immersion, $p \in M^n$. Denote by $\lambda_1, \dots, \lambda_n$ the principal curvatures of M^n at p and let e_1, \dots, e_n be an orthonormal basis of $T_p M$ such that $Ae_i = \lambda_i e_i$, $i = 1, \dots, n$. Then*

$$\frac{1}{2} \Delta |A|^2 = |\nabla A|^2 + n \sum_i \lambda_i \text{Hess } H(e_i, e_i) + \sum_{i < j} (\lambda_i - \lambda_j)^2 K_{ij}, \quad (3.1)$$

where $|\nabla A|$ is the norm of the covariant derivative ∇A of A , $\text{Hess } H$ is the Hessian of H and K_{ij} is the sectional curvature of M^n in the plane spanned by $\{e_i, e_j\}$.

Using (3.1) and the Gauss equation, one easily obtains

$$\frac{1}{2} \Delta |A|^2 = |\nabla A|^2 + n \sum_i \lambda_i H_{ii} + nc(|A|^2 - nH^2) + nH \text{tr}(A^3) - |A|^4, \quad (3.2)$$

where $\text{tr}(A^3)$ is the trace of A^3 and $H_{ii} = \text{Hess } H(e_i, e_i)$.

In this paper we deal with hypersurfaces with constant mean curvature H in Euclidean spaces. When working with hypersurfaces of constant mean constant, it is convenient to replace the shape operator A by the tensor field

$\phi = A - HI$. It is easy to see that ϕ is symmetric and that its eigenvalues are $\mu_i = \lambda_i - H$, $i = 1, \dots, n$, where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of A . It is easy to see that

$$\text{tr}(\phi) = 0, \quad (3.3)$$

$$|\phi|^2 = |A|^2 - nH^2, \quad (3.4)$$

$$\text{tr}(A^3) = \text{tr}(\phi^3) + 3H|\phi|^2 + nH^3, \quad (3.5)$$

where $|\phi|^2 = \text{tr}(\phi^2)$. From (3.4) one has that $|\phi| \equiv 0$ if and only if the immersion is totally umbilical.

In our proofs we will also need of the following result of Okumura [11]:

Lemma 3.2 (Okumura). *Let $\mu_1, \mu_2, \dots, \mu_n$, $n \geq 3$, be real numbers such that $\sum_i \mu_i = 0$ and $\sum_i \mu_i^2 = \beta^2$. Then*

$$-\frac{n-2}{\sqrt{n(n-1)}}\beta^3 \leq \sum_i \mu_i^3 \leq \frac{n-2}{\sqrt{n(n-1)}}\beta^3, \quad (3.6)$$

and equality holds in (3.6) if and only if at least $(n-1)$ of the μ_i are equal.

Let $f : M^n \rightarrow N_c^{n+1}$ be an isometric immersion with constant mean curvature H . From (3.2), (3.4) and (3.5), one obtains

$$\begin{aligned} \frac{1}{2}\Delta|A|^2 &= |\nabla A|^2 + nc(|A|^2 - nH^2) + nH\text{tr}(A^3) - |A|^4 \\ &= |\nabla A|^2 + nc|\phi|^2 + nH[\text{tr}(\phi^3) + 3H|\phi|^2 + nH^3] \\ &\quad - (|\phi|^2 + nH^2)^2 \\ &= |\nabla A|^2 + nH\text{tr}(\phi^3) + |\phi|^2[nc + nH^2 - |\phi|^2]. \end{aligned}$$

Hence, by Lema 3.2,

$$\frac{1}{2}\Delta|A|^2 \geq |\nabla A|^2 + |\phi|^2 \left[nc + nH^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}|H||\phi| - |\phi|^2 \right]. \quad (3.7)$$

Another important tool for the proofs of our results is the following well known result of Smyth and Xavier [14].

The principal curvature theorem. *Let M^n be a complete orientable Riemannian manifold, and $f : M^n \rightarrow \mathbb{R}^{n+1}$ be a isometric immersion such that $f(M^n)$ is not a hyperplane. Let $\Lambda \subset \mathbb{R}$ be the set of nonzero values assumed by the principal curvatures of shape operator A and let $\Lambda^\pm = \Lambda \cap \mathbb{R}^\pm$.*

(i) If $\Lambda^+ \neq \emptyset$ and $\Lambda^- \neq \emptyset$, then $\inf \Lambda^+ = 0 = \sup \Lambda^-$.

(ii) If Λ^+ or Λ^- is empty, then $\bar{\Lambda}$ is connected.

We will finish this section stating a result that classifies the complete isoparametric hypersurfaces in an Euclidean space. Recall that a hypersurface of the complete simply-connected $(n + 1)$ -dimensional space form \mathbb{Q}_c^{n+1} of constant sectional curvature c is called isoparametric when its principal curvatures are constant. The classification of such hypersurfaces in the case $c = 0$ was established by Levi-Civita [8] for $n = 2$ and by Segre [13] for higher dimensions. By their works, one has the following result:

Theorem 3.3. *If M^n is a complete connected Riemannian manifold and $f : M^n \rightarrow \mathbb{R}^{n+1}$ an isoparametric isometric immersion, then $f(M^n)$ is a generalized cylinder $\mathbb{R}^{n-k} \times \mathbb{S}^k(r)$, for some $k = 0, \dots, n$ and some $r > 0$.*

4 Proof of Theorem 1.1

Choose the orientation so that $H > 0$. Denoting by $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \lambda_4$ the principal curvatures of M^4 , one has, at each point of M^4 ,

$$\lambda_4 \geq H > 0. \quad (4.1)$$

Since, by hypothesis, H and R are constant, it follows from (2.8) that $|A|^2$ is constant. Hence, by (2.2) and Proposition 3.1,

$$\begin{aligned} 0 = \frac{1}{2} \Delta |A|^2 &= |\nabla A|^2 + \sum_{i < j} (\lambda_i - \lambda_j)^2 \lambda_i \lambda_j \\ &\geq \sum_{i < j} (\lambda_i - \lambda_j)^2 \lambda_i \lambda_j. \end{aligned} \quad (4.2)$$

Claim. $\inf |\lambda_3| > 0$.

Suppose, by contradiction, that there is a sequence (p_k) in M such that $\lambda_3(p_k) \rightarrow 0$. Since the principal curvature functions are bounded (because $|A|$ is constant), passing to a subsequence if necessary, we can assume that $\lambda_i(p_k) \rightarrow \bar{\lambda}_i$, $\forall i$. By (4.1),

$$\bar{\lambda}_1 \leq \bar{\lambda}_2 \leq \bar{\lambda}_3 = 0 < H \leq \bar{\lambda}_4. \quad (4.3)$$

Since H and R are constant, one has

$$\begin{aligned} 4H &= \bar{\lambda}_1 + \bar{\lambda}_2 + \bar{\lambda}_4 \\ 6R &= \bar{\lambda}_1\bar{\lambda}_2 + \bar{\lambda}_1\bar{\lambda}_4 + \bar{\lambda}_2\bar{\lambda}_4. \end{aligned}$$

Multiplying the first of the equalities above by $\bar{\lambda}_2$ and using the second, we obtain, with the aid of (4.3),

$$4H\bar{\lambda}_2 = \bar{\lambda}_1\bar{\lambda}_2 + \bar{\lambda}_2^2 + \bar{\lambda}_2\bar{\lambda}_4 = 6R - \bar{\lambda}_1\bar{\lambda}_4 + \bar{\lambda}_2^2 \geq 6R > 0,$$

whence one obtains $\bar{\lambda}_2 > 0$. Since this contradicts (4.3), the claim is proved.

It follows from the claim that $\sup \lambda_3 < 0$ or $\inf \lambda_3 > 0$. If we had $\sup \lambda_3 < 0$, then $\sup \Lambda^- = \sup \lambda_3 < 0$, contradicting the principal curvature theorem. Hence,

$$\inf \lambda_3 > 0. \tag{4.4}$$

Assume first that there is a sequence (p_k) in M such that $\lambda_2(p_k) \rightarrow 0$. Passing to a subsequence, we can assume that $\lambda_i(p_k) \rightarrow \bar{\lambda}_i$, $\forall i$. By (4.4),

$$\bar{\lambda}_1 \leq \bar{\lambda}_2 = 0 < \bar{\lambda}_3 \leq \bar{\lambda}_4. \tag{4.5}$$

Since H and R are constant, one has

$$\begin{aligned} 4H &= \bar{\lambda}_1 + \bar{\lambda}_3 + \bar{\lambda}_4, \\ 6R &= \bar{\lambda}_1\bar{\lambda}_3 + \bar{\lambda}_1\bar{\lambda}_4 + \bar{\lambda}_3\bar{\lambda}_4. \end{aligned} \tag{4.6}$$

Multiplying the first of the equalities above by $\bar{\lambda}_4$ and using the second, we obtain

$$4H\bar{\lambda}_4 = \bar{\lambda}_1\bar{\lambda}_4 + \bar{\lambda}_3\bar{\lambda}_4 + \bar{\lambda}_4^2 = 6R - \bar{\lambda}_1\bar{\lambda}_3 + \bar{\lambda}_4^2.$$

It follows from (4.5), the above equality and the hypothesis $R \geq \frac{2}{3}H^2$ that

$$\begin{aligned} 0 \geq \bar{\lambda}_1\bar{\lambda}_3 &= \bar{\lambda}_4^2 - 4H\bar{\lambda}_4 + 6R &= (\bar{\lambda}_4 - 2H)^2 - 4H^2 + 6R \\ &= (\bar{\lambda}_4 - 2H)^2 + 6\left(R - \frac{2}{3}H^2\right) \\ &\geq 0, \end{aligned}$$

and so

$$\bar{\lambda}_1 = 0, \quad \bar{\lambda}_4 = 2H \quad \text{e} \quad R = \frac{2}{3}H^2. \quad (4.7)$$

Hence, by (4.5), (4.6) and (4.7),

$$\bar{\lambda}_1 = \bar{\lambda}_2 = 0 \quad \text{e} \quad \bar{\lambda}_3 = \bar{\lambda}_4 = 2H.$$

Suppose now that $\inf |\lambda_2| > 0$. By (4.4) and the principal curvature theorem, one has

$$\inf \lambda_2 > 0. \quad (4.8)$$

We have two possibilities:

- 1) There exists some $p \in M$ such that $\lambda_1(p) = 0$.
- 2) $\lambda_1(x) \neq 0, \forall x \in M$.

Assuming 1), one has, by (4.2) and (4.8),

$$\lambda_1(p) = 0, \quad \lambda_2(p) = \lambda_3(p) = \lambda_4(p) = \frac{4}{3}H. \quad (4.9)$$

Hence, by (2.7), (2.8) and (3.4),

$$|A|^2 = \frac{16}{3}H^2, \quad R = \frac{8}{9}H^2 \quad \text{e} \quad |\phi|^2 = \frac{4}{3}H^2.$$

Substituting in (3.7), we arrive at

$$\begin{aligned} 0 = \frac{1}{2} \triangle |A|^2 &\geq |\nabla A|^2 + |\phi|^2 \left(4H^2 - \frac{4}{\sqrt{3}}H|\phi| - |\phi|^2 \right) \\ &= |\nabla A|^2 + |\phi|^2 \left(4H^2 - \frac{8}{3}H^2 - \frac{4}{3}H^2 \right) \\ &= |\nabla A|^2. \end{aligned}$$

Hence, $\nabla A \equiv 0$ and therefore f is isoparametric [12, p. 254]. Now it follows from (4.9) and Theorem 3.3 that

$$f(M^4) = \mathbb{R} \times \mathbb{S}^3 \left(\frac{3}{4H} \right).$$

Assuming 2), one has $\lambda_1 > 0$ along M . Indeed, if we had $\lambda_1(q) < 0$ at some point $q \in M$, then by continuity λ_1 would be negative along M , and from (4.8) one would obtain $\inf \Lambda^+ > 0$, contradicting the principal curvature theorem. From $\lambda_1 > 0$ and (4.2), we obtain

$$\lambda_1 \equiv \lambda_2 \equiv \lambda_3 \equiv \lambda_4 \equiv H, \quad (4.10)$$

that is, f is totally umbilical. It now follows from the classification of umbilical hypersurfaces in an Euclidean space that $f(M)$ is contained in a hypersphere or in a hyperplane of \mathbb{R}^5 . Since M is complete and $H > 0$, $f(M)$ is a hypersphere of \mathbb{R}^5 . Hence, by (4.10), $R = H^2$ and

$$f(M^4) = \mathbb{S}^4\left(\frac{1}{H}\right). \quad \square$$

5 Proof of Theorem 1.2

Choose the orientation so that $H > 0$. Denoting by $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \lambda_4 \leq \lambda_5$ the principal curvatures of M^5 , one has, at each point of M^5 ,

$$\lambda_5 \geq H > 0. \quad (5.1)$$

Since, by hypothesis, H and R are constant, it follows from (2.8) that $|A|^2$ is constant. Hence, by (2.2) and Proposition 3.1,

$$\begin{aligned} 0 = \frac{1}{2} \Delta |A|^2 &= |\nabla A|^2 + \sum_{i < j} (\lambda_i - \lambda_j)^2 \lambda_i \lambda_j \\ &\geq \sum_{i < j} (\lambda_i - \lambda_j)^2 \lambda_i \lambda_j. \end{aligned} \quad (5.2)$$

By (2.3), (2.4) and (2.6), one has

$$10R = S_2 = \lambda_i \sigma_1(\hat{\lambda}_i) + \sigma_2(\hat{\lambda}_i) = \lambda_i(5H - \lambda_i) + \sigma_2(\hat{\lambda}_i), \quad \forall i.$$

From the above equality and the hypothesis $R \geq \frac{5}{8}H^2$, one obtains

$$\begin{aligned} \sigma_2(\hat{\lambda}_i) &= \lambda_i^2 - 5H\lambda_i + 10R \\ &= \left(\lambda_i - \frac{5H}{2}\right)^2 - \frac{25}{4}H^2 + 10R \\ &= \left(\lambda_i - \frac{5H}{2}\right)^2 + 10\left(R - \frac{5}{8}H^2\right) \\ &\geq 0, \quad \forall i. \end{aligned} \quad (5.3)$$

Claim. $\inf |\lambda_4| > 0$.

Suppose, by contradiction, that there is a sequence (p_k) in M such that $\lambda_4(p_k) \rightarrow 0$. Since the principal curvature functions are bounded (because $|A|$ is constant), passing to a subsequence if necessary, we can assume that $\lambda_i(p_k) \rightarrow \bar{\lambda}_i$, $\forall i$. By (5.1),

$$\bar{\lambda}_1 \leq \bar{\lambda}_2 \leq \bar{\lambda}_3 \leq \bar{\lambda}_4 = 0 < H \leq \bar{\lambda}_5. \quad (5.4)$$

Since $\sigma_4(\bar{\lambda}_1, \bar{\lambda}_2, \bar{\lambda}_3, \bar{\lambda}_4, \bar{\lambda}_5) \geq 0$ (because, by hypothesis, $H_4 \geq 0$), one has $\bar{\lambda}_3 = 0$ and therefore,

$$0 < 10R = \bar{\lambda}_1 \bar{\lambda}_2 + \bar{\lambda}_1 \bar{\lambda}_5 + \bar{\lambda}_2 \bar{\lambda}_5 \leq \bar{\lambda}_1 \bar{\lambda}_2,$$

where in the last inequality we used (5.4). Hence,

$$\bar{\lambda}_1 \leq \bar{\lambda}_2 < \bar{\lambda}_3 = \bar{\lambda}_4 = 0 < H \leq \bar{\lambda}_5, \quad (5.5)$$

which implies

$$\lim_{k \rightarrow \infty} S_3(p_k) = \sigma_3(\bar{\lambda}_1, \bar{\lambda}_2, \bar{\lambda}_3, \bar{\lambda}_4, \bar{\lambda}_5) = \bar{\lambda}_1 \bar{\lambda}_2 \bar{\lambda}_5 > 0. \quad (5.6)$$

On the other hand, by (2.6), (5.3) and (5.5), one has

$$\begin{aligned} \lim_{k \rightarrow \infty} S_3(p_k) &= \bar{\lambda}_1 \sigma_2(\bar{\lambda}_2, \bar{\lambda}_3, \bar{\lambda}_4, \bar{\lambda}_5) + \sigma_3(\bar{\lambda}_2, \bar{\lambda}_3, \bar{\lambda}_4, \bar{\lambda}_5) \\ &\leq \sigma_3(\bar{\lambda}_2, \bar{\lambda}_3, \bar{\lambda}_4, \bar{\lambda}_5) \\ &= 0, \end{aligned}$$

contradicting (5.6). This proves the claim.

From the claim and the principal curvature theorem we obtain

$$\inf \lambda_4 > 0. \quad (5.7)$$

Assume first that there is a sequence (p_k) in M such that $\lambda_3(p_k) \rightarrow 0$. Passing to a subsequence, we can assume that $\lambda_i(p_k) \rightarrow \bar{\lambda}_i$, $\forall i$. By (5.7),

$$\bar{\lambda}_1 \leq \bar{\lambda}_2 \leq \bar{\lambda}_3 = 0 < \bar{\lambda}_4 \leq \bar{\lambda}_5, \quad (5.8)$$

and by (5.2),

$$0 \geq \sum_{i < j} (\bar{\lambda}_i - \bar{\lambda}_j)^2 \bar{\lambda}_i \bar{\lambda}_j. \quad (5.9)$$

From (2.6), (5.3) and (5.8), one obtains

$$\begin{aligned}\lim_{k \rightarrow \infty} S_3(p_k) &= \bar{\lambda}_5 \sigma_2(\bar{\lambda}_1, \bar{\lambda}_2, \bar{\lambda}_3, \bar{\lambda}_4) + \sigma_3(\bar{\lambda}_1, \bar{\lambda}_2, \bar{\lambda}_3, \bar{\lambda}_4) \\ &\geq \sigma_3(\bar{\lambda}_1, \bar{\lambda}_2, \bar{\lambda}_3, \bar{\lambda}_4) = \bar{\lambda}_1 \bar{\lambda}_2 \bar{\lambda}_4 \geq 0.\end{aligned}\tag{5.10}$$

On the other hand,

$$\begin{aligned}\lim_{k \rightarrow \infty} S_3(p_k) &= \bar{\lambda}_2 \sigma_2(\bar{\lambda}_1, \bar{\lambda}_3, \bar{\lambda}_4, \bar{\lambda}_5) + \sigma_3(\bar{\lambda}_1, \bar{\lambda}_3, \bar{\lambda}_4, \bar{\lambda}_5) \\ &\leq \sigma_3(\bar{\lambda}_1, \bar{\lambda}_3, \bar{\lambda}_4, \bar{\lambda}_5) = \bar{\lambda}_1 \bar{\lambda}_4 \bar{\lambda}_5 \leq 0.\end{aligned}\tag{5.11}$$

It follows from (5.8), (5.10) and (5.11) that $\bar{\lambda}_1 = 0$. Hence, by (5.8) and (5.9),

$$\bar{\lambda}_1 = \bar{\lambda}_2 = \bar{\lambda}_3 = 0 \quad \text{e} \quad \bar{\lambda}_4 = \bar{\lambda}_5 = \frac{5H}{2},$$

and so

$$R = \frac{5}{8}H^2.$$

Suppose now that $\inf |\lambda_3| > 0$. By (5.7) and the principal curvature theorem, one has

$$\inf \lambda_3 > 0.\tag{5.12}$$

Assume that there is a sequence (p_k) in M such that $\lambda_2(p_k) \rightarrow 0$. Passing to a subsequence, we can assume that $\lambda_i(p_k) \rightarrow \bar{\lambda}_i$, $\forall i$. By (5.12),

$$\bar{\lambda}_1 \leq \bar{\lambda}_2 = 0 < \bar{\lambda}_3 \leq \bar{\lambda}_4 \leq \bar{\lambda}_5.\tag{5.13}$$

From (5.2), we obtain

$$0 \geq \sum_{i < j} (\bar{\lambda}_i - \bar{\lambda}_j)^2 \bar{\lambda}_i \bar{\lambda}_j.\tag{5.14}$$

Since $\sigma_4(\bar{\lambda}_1, \bar{\lambda}_2, \bar{\lambda}_3, \bar{\lambda}_4, \bar{\lambda}_5) \geq 0$, one has $\bar{\lambda}_1 = 0$. It follows from (5.13) and (5.14) that

$$\bar{\lambda}_1 = \bar{\lambda}_2 = 0 \quad \text{e} \quad \bar{\lambda}_3 = \bar{\lambda}_4 = \bar{\lambda}_5 = \frac{5}{3}H,$$

and so

$$R = \frac{5}{6}H^2.$$

Assume now that $\inf |\lambda_2| > 0$. By (5.12) and the principal curvature theorem, one has

$$\inf \lambda_2 > 0. \quad (5.15)$$

We have two possibilities:

- 1) There exists some $p \in M$ such that $\lambda_1(p) = 0$.
- 2) $\lambda_1(x) \neq 0, \forall x \in M$.

Assuming 1), one has, by (5.2) and (5.15),

$$\lambda_1(p) = 0, \quad \lambda_2(p) = \lambda_3(p) = \lambda_4(p) = \lambda_5(p) = \frac{5}{4}H. \quad (5.16)$$

Hence, by (2.7), (2.8) and (3.4),

$$|A|^2 = \frac{25}{4}H^2, \quad R = \frac{15}{16}H^2 \quad \text{e} \quad |\phi|^2 = \frac{5}{4}H^2.$$

Substituting in (3.7), we arrive at

$$\begin{aligned} 0 = \frac{1}{2}\Delta|A|^2 &\geq |\nabla A|^2 + |\phi|^2 \left(5H^2 - \frac{15}{2\sqrt{5}}H|\phi| - |\phi|^2 \right) \\ &= |\nabla A|^2 + |\phi|^2 \left(5H^2 - \frac{15}{4}H^2 - \frac{5}{4}H^2 \right) \\ &= |\nabla A|^2. \end{aligned}$$

Hence, $\nabla A \equiv 0$ and therefore f is isoparametric [12, p. 254]. It now follows from (5.16) and Theorem 3.3 that

$$f(M^5) = \mathbb{R} \times \mathbb{S}^4 \left(\frac{4}{5H} \right).$$

Assuming 2), one has $\lambda_1 > 0$ along M . Indeed, if we had $\lambda_1(q) < 0$ at some point $q \in M$, then by continuity λ_1 would be negative along M , and from (5.15) one would obtain $\inf \Lambda^+ > 0$, contradicting the principal curvature theorem. From $\lambda_1 > 0$ and (5.2), we obtain

$$\lambda_1 \equiv \lambda_2 \equiv \lambda_3 \equiv \lambda_4 \equiv \lambda_5 \equiv H, \quad (5.17)$$

that is, f is totally umbilical. It now follows from the classification of umbilical hypersurfaces in an Euclidean space that $f(M)$ is contained in a hypersphere or in a hyperplane of \mathbb{R}^6 . Since M is complete and $H > 0$, $f(M)$ is a hypersphere of \mathbb{R}^6 . Hence, by (5.17), $R = H^2$ and

$$f(M^5) = \mathbb{S}^5\left(\frac{1}{H}\right). \quad \square$$

6 Hypersurfaces of arbitrary dimension

In the proof of Theorem 1.3 we will use the following lemma:

Lemma 6.1. *Let M^n be a orientable Riemannian manifold M^n , with dimension $n \geq 3$, and let $f : M^n \rightarrow N_c^{n+1}$ be an isometric immersion. Denote by A the shape operator of the immersion with respect to a global unit normal vector ξ . Then,*

$$\operatorname{tr} A^3 = \frac{nH}{2}(3|A|^2 - n^2H^2) + 3S_3. \quad (6.1)$$

Proof. It is well known (see, for example, [1, Lema 2.1.]) that

$$\operatorname{tr}(AP_r) = (r+1)S_{r+1}, \quad 1 \leq r \leq n-1,$$

where P_r is the r -th Newton's tensor (cf. Section 2). Making $r = 2$ in the above equality, one has, by (2.9),

$$3S_3 = \operatorname{tr}(AP_2) = \operatorname{tr}(S_2A - S_1A^2 + A^3) = S_2nH - S_1|A|^2 + \operatorname{tr} A^3. \quad (6.2)$$

By (2.3), (2.4) and (2.8), we have

$$n^2H^2 = |A|^2 + 2S_2. \quad (6.3)$$

From (6.2) and (6.3), we obtain

$$\begin{aligned} \operatorname{tr} A^3 &= -nH\left(\frac{n^2H^2}{2} - \frac{|A|^2}{2}\right) + nH|A|^2 + 3S_3 \\ &= \frac{nH}{2}(3|A|^2 - n^2H^2) + 3S_3. \quad \square \end{aligned}$$

Proof of Theorem 1.3. Since, by hypothesis, H and R are constant, one has by (2.8) that $|A|^2$ is also constant. Hence, by (3.2) and Lemma 6.1 one has, since $HH_3 \geq 0$ and $0 \leq R \leq \frac{nH^2}{2(n-1)}$ by hypothesis,

$$\begin{aligned}
0 = \frac{1}{2} \Delta |A|^2 &= |\nabla A|^2 + \frac{n^2 H^2}{2} (3|A|^2 - n^2 H^2) + 3nHS_3 - |A|^4 \\
&= |\nabla A|^2 - \left[|A|^4 - \frac{3n^2 H^2}{2} |A|^2 + \frac{(n^2 H^2)^2}{2} \right] + 3nHS_3 \\
&= |\nabla A|^2 - \left(|A|^2 - \frac{n^2 H^2}{2} \right) \left(|A|^2 - n^2 H^2 \right) + 3nHS_3 \\
&= |\nabla A|^2 + n^2(n-1)^2 R \left(\frac{nH^2}{2(n-1)} - R \right) + 3nHS_3 \\
&\geq |\nabla A|^2.
\end{aligned} \tag{6.4}$$

Therefore, $\nabla A \equiv 0$ and $R = 0$ or $R = \frac{nH^2}{2(n-1)}$. From $\nabla A \equiv 0$ one concludes that f is isoparametric [12, p. 254]. Since $n \geq 3$ and $H \neq 0$, it follows from Theorem 3.3 that

$$f(M^n) = \mathbb{R}^k \times \mathbb{S}^{n-k}(r),$$

for some $r > 0$ and some $1 \leq k \leq n-1$. Using (2.8), it can be easily verified that $R = 0$ occurs precisely when $k = n-1$, and $R = \frac{nH^2}{2(n-1)}$ when $k = n-2$. Hence, $f(M^n) = \mathbb{R}^{n-1} \times \mathbb{S}^1\left(\frac{1}{n|H|}\right)$ when $R = 0$, and $f(M^n) = \mathbb{R}^{n-2} \times \mathbb{S}^2\left(\frac{2}{n|H|}\right)$ when $R = \frac{nH^2}{2(n-1)}$. \square

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